Non-Abelian Painlevé equations and their monodromy surfaces

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Abstract. We discuss a connection between different linearizations for non-abelian analogs of the second Painlevé equation. For one of the non-abelian analogs, we derive the corresponding non-abelian generalizations of the monodromy surfaces.

Keywords. Non-abelian ODEs, Painlevé equations, isomonodromic Lax pairs, monodromy surfaces.

Introduction

In recent years, quantum, or more generally, non-abelian extensions of various integrable systems have acquired considerable attention. It was motivated by problematics and needs of modern quantum physics as well as by a natural attempts of mathematicians to extend and to generalize the "classical" integrable structures and systems. In particular, the Painlevé transcendents provide a good example of this phenomena. Some examples of integrable non-abelian Painlevé systems are contained in [Kaw15], [BS98], [AS21], [BS22], [RR10], [Adl20], [AK22]. Some of them were found using the existence of special isomonodromy representations or the Painlevé-Kovalevskaya test while several of the systems have been derived from integrable PDEs and lattices by reductions.

The famous Painlevé equations have being studied in various branches of mathematics and mathematical physics and have important properties. It is natural to generalize these properties to the non-abelian case. Here we are interested in non-abelian generalizations of the well-known monodromy surfaces related to different linearizations of the non-commutative analogs for the second Painlevé equation, obtained in [AS21] and labeled by P_2^0, P_2^1, and P_2^2. Below we will briefly present the results from the paper [Bob23]

Setting

We would like to work with an algebra $\hat{\mathcal{A}}$ formed by the generators $x_1^{\pm 1}, \ldots, x_N^{\pm 1}$ that is a non-abelian generalization of the Laurent polynomials in the variables $x_1, \ldots, x_N$ depending on the variables t_l, $l \in \mathbb{N}$. All elements t_l belong to a center $\mathcal{Z}(\hat{\mathcal{A}})$ of the algebra.

One can define a derivation d_{t_l} on $\hat{\mathcal{A}}$. Note that for any element $F \in \hat{\mathcal{A}}$, the element $d_{t_l}(F)$ is uniquely determined by the Leibniz rule. We use the notation " $'$" instead of d_z.

Consider the system of non-abelian ODEs

$$d_{t_l}(x_k) = F_k, \qquad F_k \in \hat{\mathcal{A}}, \qquad k = 1, \dots, N. \tag{1}$$

If for some k the element F_k depends on t_l explicitly, the system is *non-autonomous*, otherwise – *autonomous*.

Results

In the commutative case, the P_2 equation has two monodromy surfaces of the FN-type and JM-types. The corresponding linearizations are related to each other by a generalized Laplace transformation [JKT09], while the FN-type and HTW-type pairs are just gauge-equivalent [KH99] by the Fabry-type map.

We extend the list of the HTW-type pairs by those obtained in the paper [BS22] by the limiting transitions of the corresponding Lax pairs for the matrix P_4-type systems. They are given by the HTW_2^0 pair for P_2^0, $\mathrm{HTW'}_2^1$ and HTW_2^1 for P_2^1, and HTW_2^2 for the P_2^2 system. To derive the FN-type pairs from them, one is able to use the same Fabry-type map.

Note that the generalized Laplace transformation can be also extended to the non-abelian case, if we assume that one is able to eliminate terms which arise from the integration-by-parts. But, actually, that is an extra problem to prove that such a contour can be chosen. We leave this issue for a further research. Let all spectral parameters[1] be elements of $\mathcal{Z}(\hat{\mathcal{A}})$.

Proposition 1. *Let functions $W(\mu, z)$ and $Y(\lambda, z)$ be solutions of a linear problem of the form*

$$\begin{cases} \partial_\lambda \Phi(\lambda, z) &=& \mathbf{A}(\lambda, z)\, \Phi(\lambda, z), \\ \partial_z \Phi(\lambda, z) &=& \mathbf{B}(\lambda, z)\, \Phi(\lambda, z) \end{cases} \tag{2}$$

of the HTW*-type and* JM*-type, respectively. Then the* HTW*-type pair is equivalent to the* JM*-type pair by a non-abelian analog of the generalized Laplace transformation*:

$$W(\mu, z) = \int_L e^{\lambda\, \mu}\, Y(\lambda, z)\, d\lambda. \tag{3}$$

Thanks to the proposition above, we suggest a method how to construct in the non-abelian case the JM-type pairs from the HTW-pairs. It turns out that the P_2^0 system

[1] λ, μ, ζ.

has a polynomial JM-type pair, the JM_2^0 pair, which can be generalized to a fully non-commutative case. The P_2^1 system has polynomial and non-polynomial JM-type pairs (pairs $\mathrm{JM'}_2^1$ and JM_2^1, respectively). In the case of the P_2^2 system, the JM-pair is degenerate[2] (see the JM_2^2 pair). As a result, we present for each of the non-abelian P_2 systems linearizations of the HTW, FN, and JM types.

Proposition 2. *The non-abelian* P_2^0, P_2^1, and P_2^2 *systems possess linearizations of the* HTW, FN, *and* JM *types.*

To proceed to the non-abelian monodromy surfaces, we need to present a formal solution near a singular point. The following proposition is derived for this purpose.

Proposition 3. *Set* $r \geq 0$[3] *and* $\lambda \in \mathcal{Z}(\hat{\mathcal{A}})$. *Let us consider* $n \times n$*-matrices* $A(\lambda)$, $F(\lambda)$, $D(\lambda)$, $T(\lambda)$ *of the form*

$$A(\lambda) = \sum_{k \geq -r} A_k \lambda^{-k-1}, \qquad A_{-r} = diag(\alpha_1, \ldots, \alpha_n), \quad \alpha_i \neq \alpha_j, \qquad i \neq j, \tag{4}$$

$$F(\lambda) = \mathbf{I} + \sum_{k \geq 1} F_k \lambda^{-k}, \qquad D(\lambda) = \mathbf{I} + \sum_{k \geq 1} D_k \lambda^{-k}, \qquad \partial_\lambda T(\lambda) = \sum_{k=-r}^{0} T_k \lambda^{-k-1}, \tag{5}$$

where

(a) $A(\lambda) \in \mathrm{Mat}_n(\hat{\mathcal{A}})$ *and* $A_{-r} \in \mathrm{Mat}_n(\mathcal{Z}(\hat{\mathcal{A}}))$;

(b) $F_k \in \mathrm{Mat}_n(\hat{\mathcal{A}})$, $k \geq 1$, *are off-diagonal matrices*;

(c) $D_k \in \mathrm{Mat}_n(\hat{\mathcal{A}})$, $k \geq 1$, *are diagonal matrices*;

(d) $T_k \in \mathrm{Mat}_n(\mathcal{Z}(\hat{\mathcal{A}}))$, $k = -r, \ldots, 1$, *and* $T_0 \in \mathrm{Mat}_n(\hat{\mathcal{A}})$ *are both diagonal matrices*;

and suppose that

(e) *the operator* $(k\,\mathbf{I} + \mathrm{ad}_{T_0}) : \mathrm{Mat}_n(\hat{\mathcal{A}}) \to \mathrm{Mat}_n(\hat{\mathcal{A}})$, $k \geq 1$, *is invertible.*

Then the system

$$\partial_\lambda \Phi(\lambda) = A(\lambda)\, \Phi(\lambda) \tag{6}$$

admits a unique[4] *formal solution near an irregular singular point* $\lambda = \infty$ *that can be written as*

$$\Phi_{form}(\lambda) = F(\lambda)\, D(\lambda)\, \exp\left(\sum_{k=1}^{r} \tfrac{1}{k} T_{-k} \lambda^k + \ln(\lambda)\; T_0 \right) \qquad as \qquad \lambda \to \infty. \tag{7}$$

[2]We follow the terminology suggested in [JKT07], [JKT09].

[3]r is called *the Poincaré rank* of an irregular singular point. When $r = 0$, the singular point is Fuchsian.

[4]Up to a conjugation by a non-singular matrix $G \in \mathrm{Mat}_n(\hat{\mathcal{A}})$.

The proposition generalizes Proposition 2.2 in [JM81] to the non-commutative case, whose particular case was discussed in the paper [BCR18]. As a result, the formal solutions of the HTW- and JM-types near infinity were constructed.

Regarding the case of the P_2^0 system, the monodromy surfaces related to the FN_2^0 and JM_2^0 pairs are given in the following

Proposition 4. *Let x_i, $i = 1, 2, 3$, and q belong to $\hat{\mathcal{A}}$, and α, $\theta \in \mathbb{C}$. Then the monodromy surfaces related to the FN_2^0 and JM_2^0 pairs are given by the equations*

$$x_1\, x_2\, x_3 + x_1 + x_2 + x_3 - 2\sin(\pi\,\theta)\, q^{-1} = 0, \tag{8}$$

$$x_1\, x_2\, x_3 - x_1 - x_2\,(\alpha\, q^2) - x_3 + (1+\alpha)\, q^2 = 0, \tag{9}$$

respectively.

In the commutative setting, $q = 1$ and, thus, the relations above become the well-known affine cubics for the second Painlevé equation. Note also that in the commutative case these equations are equivalent by a simple scaling that cannot be generalized to the non-abelian setting. Regarding the remaining systems P_2^1 and P_2^2, the monodromy data are not isomonodromic and, thus, we cannot parameterize their solutions by the Stokes multipliers. But, in fact, one can ask about a gauge-transformation that makes the monodromy data isomonodromic. As far as we know, such a transformation does not exist.

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